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The indentation of a half-space of hexagonal elastic material by a circular punch of arbitrary end-profile

J. T. GUIDERA and R. W. LARDNER

Department of Mathematics, Simon Fraser University, Burnaby V5A 1S6, British Columbia, Canada

G. E. TUPHOLME

School of Mathematics, University of Bradford, Bradford, England

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SUMMARY
The problem is considered of the indentation by a smooth rigid punch of a half-space composed of linear elastic material of hexagonal symmetry whose plane boundary is parallel to the basal planes. The case is considered in which the area of contact between the punch and the half-space is circular, the end of the punch with is in contact with the half-space having an arbitrary profile. An integral equation is formulated and solved for the boundary value of the normal displacement in the half-space, and an expression is derived for the distribution of pressure under the punch.

1. Introduction

We shall consider the indentation of an elastic medium of hexagonal symmetry by a punch of circular cross-section whose end-profile is of arbitrary shape. We shall suppose that the medium occupies a half-space, the boundary of which is a basal plane of the hexagonal structure. This boundary is assumed to be free of tractions except for the circular area over which there is contact between the punch and the medium. Taking axes in such a way that the medium occupies the region $z > 0$, we shall suppose that the area of contact is the circle $z = 0$, $r < c$, where (r, θ, z) form a system of cylindrical polar coordinates.

We shall denote by $u_i(r, \theta, z)$ and $\sigma_{ij}(r, \theta, z)$ the physical components of displacement and stress throughout the body. Then the appropriate boundary conditions satisfied on $z = 0$ are as follows.

$$\begin{aligned} &\sigma_{rz}(r, \theta, 0) = \sigma_{\theta z}(r, \theta, 0) = 0 && \text{for all } (r, \theta), \\ &\sigma_{zz}(r, \theta, 0) = 0 && \text{for } r > c, \\ &u_z(r, \theta, 0) = U(r, \theta) + U_0 && \text{for } r < c, \end{aligned} \tag{1}$$

where $U(r, \theta)$ represents the end-profile of the punch, a given quantity, and U_0 is a constant determined by the depth of penetration. It is assumed that contact occurs between the punch and medium over the whole region $r < c$.

The total indenting force on the punch is given by

$$P = \int_0^c \int_0^{2\pi} \sigma_{zz}(r, \theta, 0) r\, dr\, d\theta. \tag{2}$$

We can assume either that P is specified and that the penetration depth U_0 is determined by the solution, or that U_0 is specified and the force P is determined by the solution.

The analogous problem for an isotropic half-space has been investigated previously by Keer [1] and Guidera [2], the former using potential methods, the latter representing the problem in terms of an integral equation. Guidera also used his method to solve related problems in which the shear displacements $u_r(r, \theta, 0)$ and $u_\theta(r, \theta, 0)$ are specified over the contact region $r < c$ instead of the shear tractions being zero there. The integral equation approach to contact problems is similar to that used by Guidera and Lardner [3] to solve the problem of an arbitrarily loaded penny-shaped crack in an isotropic medium. It has been pointed out in a note by Lardner and Tupholme [4] that, by virtue of certain results concerning the properties of dislocation loops in hexagonal crystals derived by Tupholme [5], the solutions for penny-shaped cracks in isotropic media can readily be extended to solve the analogous problems in hexagonal media. In the present paper we shall take advantage of this observation to solve the indentation problem for a hexagonal half-space by extension of Guidera's integral equation method.

We should also point out that the axi-symmetric indentation problem for a transversely isotropic elastic layer has been solved using potential methods by England [6].

The integral equation method as applied to contact problems [7] consists essentially of replacing the given problem for the half-space by a new problem which concerns an infinite medium containing a Somigliana dislocation. The surface occupied by this dislocation is chosen to coincide with the plane boundary $z = 0$ of the original half-space, the components of the displacement discontinuity across this plane being denoted by $\Delta u_i(r, \theta)$. Since the plane $z = 0$ is a plane of symmetry for the hexagonal medium, it is clear from symmetry considerations that in order for the Somigliana dislocation to provide the solution to the contact problem, the discontinuities of tangential displacement, Δu_r and Δu_θ, must in fact be zero. Furthermore, the discontinuity in normal displacement is given by $\Delta u_z(r, \theta) \equiv \equiv u_z(r, \theta, 0+) - u_z(r, \theta, 0-) = 2u_z(r, \theta, 0+)$; i.e., is equal to twice the boundary values of the surface displacement for the contact problem. Consequently, Δu_z has a known value in the contact region, given by

$$\Delta u_z(r, \theta) = 2[U(r, \theta) + U_0] \qquad (r < c). \tag{3}$$

The value of this component of displacement discontinuity in the outer region $r > c$ must be determined from the integral equation which will be written down below.

The stress components caused by an arbitrary Somigliana dislocation may be derived from the Somigliana formula [8]. When the dislocation lies on the plane surface $z = 0$ in an isotropic medium, simplified formulae have been derived [3] for the limiting values of certain of the stress components on the plane $z = 0$ itself. Now it has been shown by Tupholme [5] that for a dislocation lying in the basal plane of a hexagonal medium certain of the stress components on the plane of the dislocation can be derived from the corresponding stresses in an isotropic medium simply by making appropriate replacements of the elastic constants. Combining this observation therefore with the formulae derived in

[3] enables us to write down integral formulae for the stresses of a Somigliana dislocation in the basal plane of a hexagonal material [4].

For the present contact problem, we are interested in the stress component $\sigma_{zz}(r, \theta, 0)$, and the appropriate integral formula for this quantity is as follows [3–5].

$$\sigma_{zz}(r', \theta', 0) = -\eta_n \int_0^{2\pi} \int_0^{\infty} \left\{ \Delta u_{z,r}(r,\theta) \frac{\partial}{\partial r}\left(\frac{1}{R}\right) + \frac{1}{r^2} \Delta u_{z,\theta}(r,\theta) \frac{\partial}{\partial \theta}\left(\frac{1}{R}\right) \right\} r\,dr\,d\theta \tag{4}$$

where $R^2 = r^2 + r'^2 - 2rr' \cos(\theta - \theta')$. The constant η_n is derived from the usual elastic constants (c_{pq}) of the hexagonal medium in accordance with the following equations:

$$\eta_n = (4\pi)^{-1} K_e (c_{33}/c_{11})^{\frac{1}{2}},$$

$$K_e = (\bar{c}_{11} + c_{13})\{[c_{44}(\bar{c}_{11} - c_{13})]/[c_{33}(\bar{c}_{11} + c_{13} + 2c_{44})]\}^{\frac{1}{2}},$$

$$\bar{c}_{11} = (c_{11}c_{33})^{\frac{1}{2}}.$$

Combining the integral formula (4) with the boundary conditions (1_2) then gives that

$$\int_0^{2\pi} \int_0^{\infty} \left\{ \Delta u_{z,r}(r,\theta) \frac{\partial}{\partial r}\left(\frac{1}{R}\right) + \frac{1}{r^2} \Delta u_{z,\theta}(r,\theta) \frac{\partial}{\partial \theta}\left(\frac{1}{R}\right) \right\} r\,dr\,d\theta = 0, \qquad (r' > c). \tag{5}$$

Bearing in mind that $\Delta u_z(r, \theta)$ is determined from (3) for $r < c$, we see that (5) provides an integral equation for $\Delta u_z(r, \theta)$ in the outer region $r > c$. This equation may be solved by a technique which is similar but not identical to that used [3] for the corresponding equation which arises in the penny-shaped crack problem.

2. Solution of the problem

The integral representation (4) in two variables may be reduced to a system of relationships in a single variable by expanding all the quantities involved in Fourier series in the angular variable θ. Thus we write

$$\eta_n^{-1} \sigma_{zz}(r', \theta', 0) = \tfrac{1}{2} P_0(r') + \sum_{n=1}^{\infty} [P_n(r') \cos n\theta' + Q_n(r') \sin n\theta'], \tag{6}$$

$$\Delta u_z(r, \theta) = \tfrac{1}{2} f_0(r) + \sum_{n=1}^{\infty} [f_n(r) \cos n\theta + g_n(r) \sin n\theta], \tag{7}$$

$$R^{-1} = \tfrac{1}{2} I_0(r, r') + \sum_{n=1}^{\infty} I_n(r, r') \cos n(\theta - \theta'). \tag{8}$$

When the expansions (6)–(8) are substituted into equation (4) and the integrations over θ are performed, comparison of coefficients of $\cos n\theta'$ and $\sin n\theta'$ yields the integral representations

$$P_n(r') = -\pi \int_0^\infty r^{n+1} \frac{d}{dr}\left[\frac{f_n(r)}{r^n}\right] L_n(r, r')dr \qquad (n = 0, 1, \ldots), \tag{9}$$

$$Q_n(r') = -\pi \int_0^\infty r^{n+1} \frac{d}{dr}\left[\frac{g_n(r)}{r^n}\right] L_n(r, r')dr \qquad (n = 1, 2, \ldots), \tag{10}$$

where

$$L_n(r, r') = r^n \frac{\partial}{\partial r}[r^{-n} I_n(r, r')]. \tag{11}$$

It is clear that equation (10) may be obtained from (9) by replacing f_n and P_n by g_n and Q_n respectively.

Integral equations for the Fourier coefficients $f_n(r)$ and $g_n(r)$ in $r > c$ are obtained from equations (9) and (10) by setting $P_n(r') = Q_n(r') = 0$ for $r' > c$ (to fulfil equation (5)) and by noting that $f_n(r)$ and $g_n(r)$ are specified for $r < c$. The solutions of the resulting equations follow from the discussion in the Appendix and are found to be

$$f_n(r') = \frac{2}{\pi} r'^{-n}(r'^2 - c^2)^{\frac{1}{2}} \int_0^c \frac{r^{n+1} f_n(r)dr}{(c^2 - r^2)^{\frac{1}{2}}(r'^2 - r^2)} \qquad (r' > c), \tag{12}$$

and a corresponding expression for $g_n(r')$.

Once the Fourier coefficients f_n and g_n are known, the summation in (7) may be performed to give

$$\Delta u_z(r', \theta') = \frac{1}{\pi^2}(r'^2 - c^2)^{\frac{1}{2}} \int_0^{2\pi}\int_0^c \frac{\Delta u_z(r, \theta) r\, dr\, d\theta}{(c^2 - r^2)^{\frac{1}{2}}[r'^2 + r^2 - 2rr'\cos(\theta - \theta')]} \qquad (r' > c), \tag{13}$$

which is an integral formula expressing the displacement discontinuity $\Delta u_z(r', \theta')$ in the outer region $r' > c$ in terms of its values in the region $r' < c$. Recalling that Δu_z is related to the boundary value of the normal displacement for the contact problem by the equation $\Delta u_z(r, \theta) = 2u_z(r, \theta, 0)$, we see that equations (3) and (13) determine the normal displacement on the boundary $z = 0$ completely in terms of prescribed data.

The Fourier coefficients of the normal stress component can be found by substituting the solutions for $f_n(r)$ and $g_n(r)$ into the integral representations (9) and (10). After some manipulation and integration we obtain

$$P_n(r') = 4r'^{n-1} \frac{d}{dr'} \int_{r'}^c \frac{t^{1-2n}}{(t^2 - r'^2)^{\frac{1}{2}}} \frac{d}{dt} \int_0^t \frac{r^{n+1} f_n(r)}{(t^2 - r^2)^{\frac{1}{2}}} dr\, dt \qquad (r' < c). \tag{14}$$

A similar expression is obtained relating $Q_n(r')$ to $g_n(r)$. By substituting these results into equation (6), the normal traction under the punch can be found.

Combining equations (2) and (6), we find that the total force on the punch is given by

$$P = \pi \eta_n \int_0^c P_0(r) r\, dr.$$

From equation (14) with $n = 0$ therefore we obtain that

$$P = -4\eta_n \int_0^{2\pi} \int_0^c \frac{\Delta u_z(r, \theta)}{(c^2 - r^2)^{\frac{1}{2}}} r\,dr\,d\theta. \tag{15}$$

Should the indenting force P be specified, the penetration depth, U_0, may be found by substituting (3) into (15) and solving for U_0. This gives

$$U_0 = -(P/16\pi\eta_n c) - (2\pi c)^{-1} \int_0^{2\pi} \int_0^c \frac{U(r, \theta)}{(c^2 - r^2)^{\frac{1}{2}}} r\,dr\,d\theta.$$

Appendix

The technique of solution of the integral equations which arise by combining equations (9), (10), (6) and the boundary condition (1_2) is similar to that used in [3]. Consider the equation

$$\int_0^\infty r^{n+1} \frac{d}{dr}\left[\frac{f_n(r)}{r^n}\right] L_n(r, r')dr = -\frac{1}{\pi} P_n(r')H(c - r'), \tag{A1}$$

where $H(c - r')$ is the Heaviside function. Now it has been shown [2, 3] that $L_n(r, r')$ has the integral representation

$$L_n(r, r') = -2 \int_0^\infty J_{n+1}(r\xi)J_n(r'\xi)\xi\,d\xi. \tag{A2}$$

Taking into account (A2), we notice that (A1) is a Hankel transform over r' which may be inverted to give

$$\int_0^\infty r^{n+1} \frac{d}{dr}\left[\frac{f_n(r)}{r^n}\right] J_{n+1}(r\xi)dr = \frac{1}{2\pi} \int_0^c P_n(r)J_n(r\xi)r\,dr.$$

The left-hand side is now integrated by parts; since $f_n(r)$ is known for $r < c$ we write the result in the form

$$\xi \int_0^c rf_n(r)J_n(r\xi)dr + \xi \int_c^\infty rf_n(r)J_n(r\xi)dr = -\frac{1}{2\pi} \int_0^c P_n(r)J_n(r\xi)r\,dr.$$

Multiplying both sides by $\xi^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\xi t)$, $t > c$ and integrating over ξ from 0 to ∞ we obtain

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^c \frac{r^{n+1}f_n(r)dr}{(t^2 - r^2)^{\frac{1}{2}}} + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_c^t \frac{r^{n+1}f_n(r)dr}{(t^2 - r^2)^{\frac{1}{2}}}$$

$$= -\frac{1}{2^{\frac{3}{2}}\pi} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n + 1)} \int_0^c P_n(r)r^{n+1}\,dr.$$

We next multiply both sides by $t(\rho^2 - t^2)^{-\frac{1}{2}}$ and integrate over t from c to ρ:

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_c^\rho \frac{t}{(\rho^2 - t^2)^{\frac{1}{2}}} dt \int_0^c \frac{r^{n+1} f_n(r) dr}{(t^2 - r^2)^{\frac{1}{2}}} + \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \int_c^\rho r^{n+1} f_n(r) dr$$

$$= -\frac{1}{2^{\frac{1}{2}}\pi} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} (\rho^2 - c^2)^{\frac{1}{2}} \int_0^c P_n(r) r^{n+1} dr.$$

Finally, differentiating with respect to ρ we obtain

$$\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \rho^n f_n(\rho) = -(\rho^2 - c^2)^{-\frac{1}{2}}$$

$$\times \left\{ \frac{\Gamma(n+\frac{1}{2})}{2^{\frac{1}{2}}\pi\Gamma(n+1)} \int_0^c P_n(r) r^{n+1} dr + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^c \frac{r^{n+1} f_n(r) dr}{(c^2 - r^2)^{\frac{1}{2}}} \right\}$$

$$+ \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_c^\rho \frac{t\,dt}{(t^2 - r^2)^{\frac{1}{2}}} \int_0^c \frac{r^{n+1} f_n(r) dr}{(\rho^2 - t^2)^{\frac{1}{2}}}.$$

We require the Fourier coefficients of displacement discontinuity to be bounded at $\rho = c$. In order that this be true we must have that

$$\frac{\Gamma(n+\frac{1}{2})}{2^{\frac{1}{2}}\pi\Gamma(n+1)} \int_0^c P_n(r) r^{n+1} dr = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^c \frac{r^{n+1} f_n(r) dr}{(c^2 - r^2)^{\frac{1}{2}}}.$$

Then finally,

$$f_n(\rho) = \frac{2}{\pi} \rho^{-n} \int_c^\rho \frac{t\,dt}{(t^2 - r^2)^{\frac{1}{2}}} \int_0^c \frac{r^{n+1} f_n(r) dr}{(\rho^2 - t^2)^{\frac{1}{2}}}.$$

The t-integration may be performed to give

$$f_n(\rho) = \frac{2}{\pi} \rho^{-n} (\rho^2 - c^2)^{\frac{1}{2}} \int_0^c \frac{r^{n+1} f_n(r) dr}{(c^2 - r^2)^{\frac{1}{2}} (\rho^2 - r^2)}.$$

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